Useful Adaptive Logics for Rational and Paraconsistent Reasoning

Ofer Arieli

Report CW 286, April 2000

Katholieke Universiteit Leuven
Department of Computer Science
Celestijnenlaan 200A – B-3001 Heverlee (Belgium)

Useful Adaptive Logics for Rational and Paraconsistent Reasoning

Ofer Arieli

Report CW 286, April 2000

Department of Computer Science, K.U.Leuven

Abstract

We introduce a family of logics that are particularly useful for reasoning with uncertainty. The consequence relations considered here are defined by a simple and natural semantics, and have many desirable properties. In particular, these relations are nonmonotonic, paraconsistent, adaptive in the sense of Batens, plausibility logics in the sense of Lehmann, and rational in the sense of Lehmann and Magidor.

Useful Adaptive Logics for Rational and Paraconsistent Reasoning

Ofer Arieli

Department of Computer Science
K.U.Leuven
Celestijnenlaan 200A, B-3001 Heverlee, Belgium
arieli@cs.kuleuven.ac.be

April 17, 2000

Abstract

We introduce a family of logics that are particularly useful for reasoning with uncertainty. The consequence relations considered here are defined by a simple and natural semantics, and have many desirable properties. In particular, these relations are nonmonotonic, paraconsistent, adaptive in the sense of Batens, plausibility logics in the sense of Lehmann, and rational in the sense of Lehmann and Magidor.

1 Introduction

In this work we investigate and characterise a family of consequence relations for reasoning with uncertainty. Generally, we consider two types of uncertainty:

a) The data is incomplete. In such cases only a partial information about the domain of discourse is available, and so one has to draw conclusions despite the lack of knowledge.

b) The data is inconsistent, thus one has to make inferences from a contradictory information.

It is well-known that classical logic cannot deal properly with these types of uncertainty. This is so mainly due to the fact that classically *any* formula is a logical consequence of an inconsistent theory, and so one cannot make classical inferences from inconsistent knowledge-bases in a non-trivial way. Moreover, being monotonic, classical logic cannot support default reasoning, and so its use with incomplete information is problematic as well.

We follow here a common approach to overcome the shortcomings of classical logic by turning to multiple-valued logics. Many formalisms for reasoning with many-valued semantics have been proposed in the literature, using every possible number of truth-values, from three values (see, e.g., [5] for a survey of some natural three-valued logics), up to arbitrarily many values (used e.g., in possiblistic logics [12], annotated logics [26, 27], and many formalisms that are based on fuzzy logic or probabilistic reasoning. See, for example, a survey in [22]). In most of these approaches the truth-values are arranged in a lattice structure. Lattices, and in particular a special family of them, called *logical* lattices, will be our main semantical tool here.

Our major concern here will be to recapture within this multy-valued framework classical reasoning (where its use is appropriate), as well as some standard non-monotonic and paraconsistent methods. For that, we incorporate a concept first introduced by McCarthy [21] and later by Shoham [25]. They suggested that in order to make inferences from a given theory one should consider only a subset of the models of that theory. This set of *preferential models* is determined according to some conditions that can be specified syntactically by a set of (usually second-order) propositions, or semantically by using some order relation on the models of the theory. This relation should reflect some kind of preference criterion on the models of the set of premises. In our case the idea is to give precedence to those valuations that minimize the amount of inconsistent belief in the set of premises. This approach reflects the intuition that while one has to deal with conflicts in a nontrivial way, contradictory data corresponds to inadequate information about the real world, and therefore should be minimized.

The rest of this paper is organised as follows: In next section we introduce our framework and demonstrate its usefulness by using some simple toy examples. In Section 3 we characterise the consequence relations that are induced by this framework. This allows us to consider, in Section 4, some useful properties of these relations, which make them particularly suitable for imitating commonsense reasoning. In Section 5 we compare our approach to some related formalisms, and in Section 6 we conclude.

2 Preliminaries

2.1 Logical lattices and their consequence relations

In what follows we denote by $\mathcal{L} = (L, \leq)$ a bounded lattice that has at least three elements ("truth values"): A $\leq$-maximal element and a $\leq$-minimal element that correspond to the classical values (denoted, respectively, by t and f), and an intermediate element (denoted by $\top$) that may intuitively be understood as representing contradictions. As usual, the meet and the join operations on $\mathcal{L}$ are denoted by $\wedge$ and $\vee$. In addition, we assume that $\mathcal{L}$ has an involution operator $\neg$ (a "negation") s.t. $\neg t = f$, $\neg f = t$, and $\neg\top = \top$. We denote by $\mathcal{D}$ the set of the *designated* values of $\mathcal{L}$ (i.e., the set of the truth values in $\mathcal{L}$ that represent true assertions). We shall assume that $\mathcal{D}$ is a prime filter in $\mathcal{L}$,[1] and that $\top \in \mathcal{D}$. The pair $(\mathcal{L}, \mathcal{D})$ is called a *logical lattice* [4].

The smallest logical lattice is $\mathcal{THREE}$, in which $f < \top < t$, and $\{t, \top\}$ is the set of the designated elements. It provides the semantical background to many formalisms considered in the literature (Kleene 3-valued logics with middle element designated LP [14], Priest's LPm

[1] In particular $t \in \mathcal{D}$ and $f \notin \mathcal{D}$.

[23, 24], etc.). Belnap's well-known lattice $\mathcal{FOUR}$ [9, 10] obtains by adding to the three basic elements a fourth one, $\bot$, s.t. $\bot \notin \mathcal{D}$ and $\neg\bot = \bot$. As a results, in $\mathcal{FOUR}$, beside the two classical elements t and f, there are two intermediate elements that intuitively represent the two cases of uncertainty: $\bot$ for a lack of knowledge, and $\top$ for "over"-knowledge (i.e., contradictions). In what follows we shall sometimes abbreviate "3" for $\mathcal{THREE}$ and "4" for $\mathcal{FOUR}$.

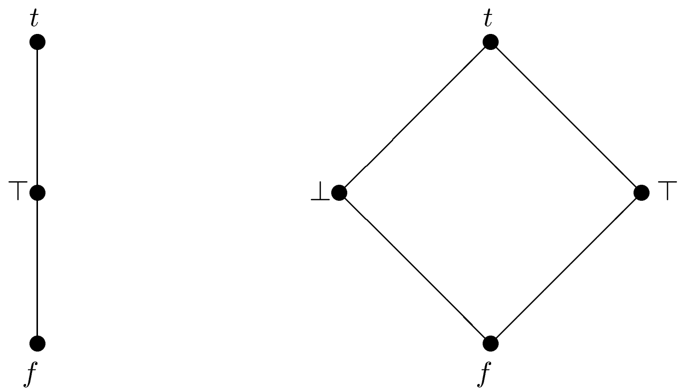

Figure 1: $\mathcal{THREE}$ and $\mathcal{FOUR}$

Given a logical lattice $(\mathcal{L}, \mathcal{D})$, the standard semantical notions are natural generalizations of the classical ones: A (multiple-valued) *valuation* ν is a function that assigns an element of L to each atomic formula. The set of valuations onto L is denoted by $\mathcal{V}^L$. Extension to complex formulae is done in the usual way:

1. $\nu(\psi \vee \phi) = \nu(\psi) \vee \nu(\phi)$,
2. $\nu(\psi \wedge \phi) = \nu(\psi) \wedge \nu(\phi)$,
3. $\nu(\neg\psi) = \neg\nu(\psi)$.

A valuation is a *model* of a set of assertions if it assigns a designated value to every formula in this set. The models of a set Γ are usually denoted by M or N (possibly with subscripts). The set of all the models of Γ is denoted by $mod(\Gamma)$.

The language that we will consider here is a propositional one. Note that there are no tautologies in the language of $\{\neg, \vee, \wedge\}$; If all the atomic formulae that appear in a formula ψ are assigned $\bot$ by a valuation ν, then $\nu(\psi) = \bot$ as well. In particular, excluded middle $(\psi \vee \neg\psi)$ is not a valid rule here. Hence, the definition of the material implication $p \leadsto q$ as $\neg p \vee q$ is not adequate for representing entailments in our multiple-valued semantics. Instead, we use another connective, which does function as an implication in our setting:

Definition 2.1 [2, 5] Let $(\mathcal{L}, \mathcal{D})$ be a logical lattice. Define: $x \supset y = y$ if $x \in \mathcal{D}$, and $x \supset y = t$ otherwise.

On $\{t, f\}$ the material implication ($\leadsto$) and the new implication ($\supset$) are identical, and both of them are generalisations of the classical implication. However, relative to the basic consequence relation of logical lattices defined in 2.2, Modus Ponens and the deduction theorem fail for $\leadsto$, while both of them are valid for $\supset$.

The language of $\{\neg, \vee, \wedge, \supset\}$ together with the propositional constants $t, f, \top, \bot$ will be denoted henceforth by Σ.[2] We shall denote by p or q atomic formulae, complex formulae in Σ are denoted by ψ or ϕ, and sets of formulae are denoted by Γ or Δ. $\mathcal{A}(\Gamma)$ denotes the set of the atomic formulae that appear in some formula of Γ.

A natural definition of a lattice-based consequence relation would now be the following:

Definition 2.2 Let $(\mathcal{L}, \mathcal{D})$ be a logical lattice. Denote $\Gamma \models^{\mathcal{L},\mathcal{D}} \Delta$ if every model of Γ is a model of some formula in Δ.

2.2 The consequence relation $\models_c^{\mathcal{L},\mathcal{D}}$

The relation $\models^{\mathcal{L},\mathcal{D}}$ of Definition 2.2 is a consequence relation in the standard sense of Tarski and Scott. In [2] it is shown that this relation is sound and complete w.r.t. a certain cut-free Gentzen-type system, and that it is also monotonic, compact, and paraconsistent [11]. The main drawbacks of $\models^{\mathcal{L},\mathcal{D}}$ are that it is strictly weaker than classical logic even for consistent theories, and that it always invalidates some intuitively justified inference rules, like the Disjunctive Syllogism ($\psi, \neg\psi \vee \phi \not\models^{\mathcal{L},\mathcal{D}} \phi$).

In what follows we therefore consider a family of consequence relations that are obtained by using a more liberal semantics: A formula ψ may be deduced from a set Γ of formulae if ψ holds in every *preferred* model (and not necessarily *all* the models) of Γ. The preferred models are determined according to some criterion for making preferences among valuations (see, e.g., [15, 19, 20, 21, 25]). We introduce here a general criterion for making such preferences: The truth values are arranged according to an order relation that reflects differences in the amount of inconsistency that each one of them exhibits. Then we choose those valuations that minimize the amount of inconsistent knowledge w.r.t. this order. This approach reflects the intuition that contradictory data corresponds to inadequate information about the real world, and therefore should be minimized. It will allow us to define a family of consequence relations with many desirable properties, some of which are considered in Section 4.

Definition 2.3 A partial order $<$ on a set L is called *modular* if $y < x_2$ for every $x_1, x_2, y \in L$ s.t. $x_1 \not< x_2$, $x_2 \not< x_1$, and $y < x_1$.

[2] In the context of $\mathcal{THREE}$, Σ will denote the language of $\{\neg, \vee, \wedge, \supset, t, f, \top\}$.

Proposition 2.4 [18] Let $<$ be a partial order on L. The following conditions are equivalent:

a) $<$ is modular.

b) For every $x_1, x_2, y \in L$, if $x_1 < x_2$ then either $y < x_2$ or $x_1 < y$.

c) There is a totally ordered set L' with a strict order $\prec$ and a function $g: L \to L'$ s.t. $x_1 < x_2$ iff $g(x_1) \prec g(x_2)$.

Definition 2.5 An *inconsistency order* $<_c^{\mathcal{L},\mathcal{D}}$ on $(\mathcal{L}, \mathcal{D})$ is a well-founded modular order on the elements of L that satisfies the following properties:

a) t and f are minimal and $\top$ is maximal w.r.t. $<_c^{\mathcal{L},\mathcal{D}}$,

b) if $\{x, \neg x\} \subseteq \mathcal{D}$ while $\{y, \neg y\} \not\subseteq \mathcal{D}$, then $x \not<_c^{\mathcal{L},\mathcal{D}} y$,

c) x and $\neg x$ are either equal or $<_c^{\mathcal{L},\mathcal{D}}$-incomparable.

Intuitively, if $x <_c^{\mathcal{L},\mathcal{D}} y$, then x represents a knowledge (or belief) that is more consistent than the knowledge that is represented by y. The reason for requiring that an inconsistency order would be modular is to disallow non-intuitive cases like $\{\{t\}, \{f <_c^{\mathcal{L},\mathcal{D}} \bot <_c^{\mathcal{L},\mathcal{D}} \top\}\}$, in which, e.g., $\top$ is incomparable with t w.r.t. the amount of inconsistency that they represent (while in this case $\top$ *is* comparable with $\neg t$). We also require that truth values that intuitively represent inconsistent belief should not be less inconsistent than those ones that reflect consistent belief. Finally, a truth value should not be strictly more or strictly less (in)consistent than its negation.

Example 2.6 $\mathcal{THREE}$ has two inconsistency orders:

a) The degenerated inconsistency order, $<_{c_0}^3$, in which $t, f, \top$ are all incomparable.

b) $<_{c_1}^3$, in which $\top$ is strictly more inconsistent than the other truth values: $\{t, f\} <_{c_1}^3 \top$.

In $\mathcal{FOUR}$ there are four inconsistency orders:

a) The degenerated inconsistency order, $<_{c_0}^4$, in which $t, f, \bot, \top$ are all incomparable.

b) $<_{c_1}^4$, in which $\bot$ is considered as minimally inconsistent: $\{t, f, \bot\} <_{c_1}^4 \top$.

c) $<_{c_2}^4$, in which $\bot$ is maximally inconsistent: $\{t, f\} <_{c_2}^4 \{\top, \bot\}$.

d) $<_{c_3}^4$, in which $\bot$ represents some intermediate level of inconsistency: $\{t, f\} <_{c_3}^4 \bot <_{c_3}^4 \top$.

In the rest of the paper we shall continue to use the notations of Example 2.6 for denoting the inconsistency orders in $\mathcal{THREE}$ and $\mathcal{FOUR}$.

Given an inconsistency order $\leq_c^{\mathcal{L},\mathcal{D}}$ in a logical lattice $(\mathcal{L}, \mathcal{D})$, it induces an equivalence relation on L, in which two elements in L are equivalent if they are equal or $\leq_c^{\mathcal{L},\mathcal{D}}$-incomparable. For every $x \in \mathcal{L}$, we denote by $[x]$ its equivalence class w.r.t. this equivalence relation. I.e.,

$$[x] = \{y \mid y = x, \text{ or } x \text{ and } y \text{ are } \leq_c^{\mathcal{L},\mathcal{D}}\text{-incomparable}\}.$$

The order relation on these classes is defined as usual by representatives: $[x] \leq_c^{\mathcal{L},\mathcal{D}} [y]$ iff either $x \leq_c^{\mathcal{L},\mathcal{D}} y$, or x and y are $\leq_c^{\mathcal{L},\mathcal{D}}$-incomparable.[3] It is easy to verify that this definition is proper, i.e. it does not depend on the choice of the representatives.

Definition 2.7 Let $\leq_c^{\mathcal{L},\mathcal{D}}$ be an inconsistency order in a logical lattice $(\mathcal{L}, \mathcal{D})$, and let $\nu_1, \nu_2 \in \mathcal{V}^L$.

a) $\nu_1 \leq_c^{\mathcal{L},\mathcal{D}} \nu_2$ iff for every atom p, $[\nu_1(p)] \leq_c^{\mathcal{L},\mathcal{D}} [\nu_2(p)]$.

b) $\nu_1 <_c^{\mathcal{L},\mathcal{D}} \nu_2$ if $\nu_1 \leq_c^{\mathcal{L},\mathcal{D}} \nu_2$ and there is an atomic formula q for which $[\nu_1(q)] <_c^{\mathcal{L},\mathcal{D}} [\nu_2(q)]$.

Definition 2.8 Let $\leq_c^{\mathcal{L},\mathcal{D}}$ be an inconsistency order in a logical lattice $(\mathcal{L}, \mathcal{D})$. The set of the *c-most consistent models* of a set Γ of formulae in Σ (abbreviation: the c-mcms of Γ) is defined as follows:

$$!(\Gamma, \leq_c^{\mathcal{L},\mathcal{D}}) = \{M \in mod(\Gamma) \mid \neg\exists N \in mod(\Gamma) \text{ s.t. } N <_c^{\mathcal{L},\mathcal{D}} M\}.$$

Now we can refine the inference mechanism imposed by the lattice-based consequence relation $\models^{\mathcal{L},\mathcal{D}}$ of Definition 2.2; Instead of considering every possible model of the premises, now we should consider only the c-most consistent ones. As we shall see in Section 4, by doing so we get more subtle consequence relations.

Definition 2.9 Let $\leq_c^{\mathcal{L},\mathcal{D}}$ be an inconsistency order in a logical lattice $(\mathcal{L}, \mathcal{D})$. Denote $\Gamma \models_c^{\mathcal{L},\mathcal{D}} \Delta$ if every c-mcm of Γ is a model of some formula in Δ.

2.3 Examples

Let us demonstrate the behaviour of $\models_c^{\mathcal{L},\mathcal{D}}$ on some well-known toy examples. First, we extend the discussion to first-order logics. It is possible to do so in a straightforward way, provided that there are no quantifiers within the formulae, and that each formula that contains variables is considered as universally quantified. Consequently, a set of assertions Γ containing a non-grounded formula, ψ, is viewed as representing the corresponding set of ground formulae formed by substituting for each variable that appears in ψ every possible element of Herbrand universe, U. Formally: $\Gamma^U = \{\rho(\psi) \mid \psi \in \Gamma,\ \rho : var(\psi) \to U\}$, where ρ is a *ground substitution* from the variables of every $\psi \in \Gamma$ to the individuals of U.

Example 2.10 (The Barber Paradox) Consider one direction of the barber paradox:

$$\Gamma = \{\neg \mathtt{shaves(x,x)} \supset \mathtt{shaves(Barber,x)}\}.$$

Denote by ν_1, ν_2, and ν_3 the valuations that assign t, $\perp$, and $\top$ (respectively) to the assertion `shaves(Barber,Barber)`. Using $\mathcal{FOUR}$ as the underlying logical lattice, we have that

$$!(\Gamma, \leq_{c2}^4) = !(\Gamma, \leq_{c3}^4) = \{\nu_1\}, \quad !(\Gamma, \leq_{c1}^4) = \{\nu_1, \nu_2\}, \quad !(\Gamma, \leq_{c0}^4) = \{\nu_1, \nu_2, \nu_3\}.$$

Thus $\Gamma \not\models_{c_i}^4$ `shaves(Barber,Barber)` in case that $i = 0, 1$, while $\Gamma \models_{c_i}^4$ `shaves(Barber,Barber)` in case that $i = 2, 3$.

[3] As usual, we use the same notation to denote the order relation among equivalence classes and the order relation among their elements.

Example 2.11 (Tweety Dilemma) Consider the following well-known knowledge-base:

$\texttt{bird(x)} \rightsquigarrow \texttt{fly(x)}$

$\texttt{penguin(x)} \supset \texttt{bird(x)}$

$\texttt{penguin(x)} \supset \neg\texttt{fly(x)}$

$\texttt{bird(Tweety)}$

$\texttt{bird(Fred)}$

We are using different implication connectives here according to the strength we attach to each entailment: Penguins *never* fly. This is a characteristic property of penguins, and there are no exceptions to that. Also, every penguin is a bird, and again, there are no exceptions to that fact. Thus, the second and the third rules are formulated with a stronger implication connective than the first rule, which states only a default property of birds.

Let us consider this example in $\mathcal{FOUR}$. Denote the above set of assertions by Γ. This set has 324 ($= 18^2$) four-valued models altogether. Since the roles of Tweety and Fred are totally symmetric, we give in Table 1 only the 18 model-assignments that concern with Tweety.

Table 1: The assignments of $\langle$Predicate$\rangle$(Tweety) in $mod(\Gamma)$

Model No.	`bird(Tweety)`	`fly(Tweety)`	`penguin(Tweety)`
$M_1 - M_8$	$\top$	$\top, f$	$\top, t, f, \bot$
$M_9 - M_{12}$	$\top$	$t, \bot$	$f, \bot$
$M_{13} - M_{16}$	t	$\top$	$\top, t, f, \bot$
$M_{17} - M_{18}$	t	t	$f, \bot$

Here, $!(\Gamma, \leq^4_{c_i}) = \{M_{17}, M_{18}\}$ for $i=1$ and $!(\Gamma, \leq^4_{c_i}) = \{M_{17}\}$ for $i=2,3$. Thus, when using $\models^4_{c_i}$ for any $1 \leq i \leq 3$, one can infer from Γ that `bird(Tweety)` (but $\neg$`bird(Tweety)` is not true), and that `fly(Tweety)` (while $\neg$`fly(Tweety)` is not true). On the other hand, $\neg$`penguin(Tweety)` is deducible only by $\models^4_{c_2}$ and $\models^4_{c_3}$ (while `penguin(Tweety)` is not deducible by either one of them).

Suppose now that a new datum arrives, and Tweety is known to be a penguin. Denote the new set of assertions by Γ'. I.e.,

$$\Gamma' \;=\; \Gamma \;\cup\; \{\; \texttt{penguin(Tweety)} \;\}$$

Clearly, Γ' is no longer classically consistent. This implies that everything classically follows from it. In particular, although the conflict in Γ' has nothing to do with the information about Fred, and despite the fact that the data about Fred has not been changed, classical logic is still useless for reasoning about Fred, since every fact is now classically provable.

As it is shown in Section 4.1, consequence relations of the form $\models_c^{\mathcal{L},\mathcal{D}}$ are paraconsistent, and so they do not have this drawback; Although Γ' is classically inconsistent, nontrivial conclusions about Tweety and Fred can be obtained by $\models_{c_i}^4$. Indeed, Γ' has 6×18 four-valued models. The 18 model assignments for the predicates that concern with Fred remain the same as those of Γ (since in the information about Fred has not been changed). However, the assignments for the predicates that are related with Tweety are totally different. The six new model-assignments are listed in Table 2.

Table 2: The assignments of $\langle$Predicate$\rangle$(Tweety) in $mod(\Gamma')$

Model No.	`bird(Tweety)`	`fly(Tweety)`	`penguin(Tweety)`
$M_1 - M_2$	$\top$	$\top$	$\top, t$
$M_3 - M_4$	$\top$	f	$\top, t$
$M_5 - M_6$	t	$\top$	$\top, t$

This time, therefore, $!(\Gamma', \leq_{c_i}^4) = \{M_4, M_6\}$ for $i = 1, 2, 3$. It follows that `bird(Tweety)`, `penguin(Tweety)`, and $\neg$`fly(Tweety)` are all deducible from Γ' relative to $\models_{c_i}^4$ $(i=1,2,3)$. The complements of these assertions cannot be inferred by any one of the consequence relations, as indeed one expects.

3 Representation result

In this section we characterise the family of the consequence relations of the form $\models_c^{\mathcal{L},\mathcal{D}}$, assuming that the set $\mathcal{V}^L$ of valuations onto $(\mathcal{L}, \mathcal{D})$ is *stoppered* w.r.t. $\leq_c^{\mathcal{L},\mathcal{D}}$, i.e.: for every set of formulae Γ and every $M \in mod(\Gamma)$, either $M \in !(\Gamma, \leq_c^{\mathcal{L},\mathcal{D}})$, or there is an $M' \in !(\Gamma, \leq_c^{\mathcal{L},\mathcal{D}})$ s.t. $M' <_c^{\mathcal{L},\mathcal{D}} M$.[4]

Note that in case that $\mathcal{V}^L$ is well-founded w.r.t. $\leq_c^{\mathcal{L},\mathcal{D}}$ (i.e., $\mathcal{V}^L$ does not have an infinitely descending chain w.r.t. $\leq_c^{\mathcal{L},\mathcal{D}}$), then it is in particular stoppered.

In $\mathcal{THREE}$ there are only two consequence relations of the form $\models_c^{\mathcal{L},\mathcal{D}}$. One, $\models_{c_0}^3$, corresponds to the (degenerated) inconsistency order $<_{c_0}^3$. By its definition, it is the same as $\models^3$, and so it has the properties of consequence relations of the form $\models^{\mathcal{L},\mathcal{D}}$, mentioned at the beginning of Section 2.2. The other consequence relation, $\models_{c_1}^3$, corresponds to the inconsistency order $<_{c_1}^3$ (see Example 2.6). This consequence relation is in fact the same as that of the logic LPm of Priest [23, 24], and we consider its main properties in Section 5, where we compare LPm to our formalisms.

The following theorem characterises the families of consequence relation of the form $\models_c^{\mathcal{L},\mathcal{D}}$ for logical lattices with more than three elements. It is shown that in such cases $\mathcal{FOUR}$ is canonical:

[4]The notion "stopperdness" is due to Mackinson [20]; In [15] the same property is called *smoothness*.

Theorem 3.1 Let $(\mathcal{L},\mathcal{D})$ be a logical lattice with at least four elements, and let $\leq_c^{\mathcal{L},\mathcal{D}}$ be an inconsistency order in $(\mathcal{L},\mathcal{D})$ that induces a stoppered relation on $\mathcal{V}^L$. Then there is an inconsistency order $\leq_{c_i}^4$ $(0\leq i\leq 3)$ in $\mathcal{FOUR}$ s.t. $\Gamma\models_c^{\mathcal{L},\mathcal{D}}\Delta$ iff $\Gamma\models_{c_i}^4\Delta$.

For the proof of Theorem 3.1 and for what follows we shall need the following notations and definitions:

Notation 3.2 Given a logical lattice $(\mathcal{L},\mathcal{D})$, its elements may be divided into the following four sets:

$$\mathcal{T}_t^{\mathcal{L},\mathcal{D}}=\{x\in L \mid x\in\mathcal{D},\neg x\notin\mathcal{D}\},\qquad \mathcal{T}_f^{\mathcal{L},\mathcal{D}}=\{x\in L \mid x\notin\mathcal{D},\neg x\in\mathcal{D}\},$$
$$\mathcal{T}_\top^{\mathcal{L},\mathcal{D}}=\{x\in L \mid x\in\mathcal{D},\neg x\in\mathcal{D}\},\qquad \mathcal{T}_\perp^{\mathcal{L},\mathcal{D}}=\{x\in L \mid x\notin\mathcal{D},\neg x\notin\mathcal{D}\}.$$

We shall usually omit the superscripts, and just write $\mathcal{T}_t,\mathcal{T}_f,\mathcal{T}_\top$, $\mathcal{T}_\perp$.

Notation 3.3 Let $(\mathcal{L},\mathcal{D})$ be a logical lattice. Denote:

$$\min_{\leq_c^{\mathcal{L},\mathcal{D}}}\mathcal{T}_x=\{y\in\mathcal{T}_x \mid \neg\exists y'\in\mathcal{T}_x \text{ s.t. } y'<_c^{\mathcal{L},\mathcal{D}}y\}\quad (x\in\{t,f,\top,\perp\})$$
$$\Omega_{\leq_c^{\mathcal{L},\mathcal{D}}}=\min_{\leq_c^{\mathcal{L},\mathcal{D}}}\mathcal{T}_t\cup\min_{\leq_c^{\mathcal{L},\mathcal{D}}}\mathcal{T}_f\cup\min_{\leq_c^{\mathcal{L},\mathcal{D}}}\mathcal{T}_\perp\cup\min_{\leq_c^{\mathcal{L},\mathcal{D}}}\mathcal{T}_\top$$

Definition 3.4 Let $(\mathcal{L}_1,\mathcal{D}_1)$ and $(\mathcal{L}_2,\mathcal{D}_2)$ be two logical lattices. Suppose that x_i is some element of L_i and ν_i is a valuation onto L_i $(i=1,2)$.

a) x_1 and x_2 are *similar* if for some $y\in\{t,f,\top,\perp\}$ we have that $x_1\in\mathcal{T}_y^{\mathcal{L}_1,\mathcal{D}_1}$ iff $x_2\in\mathcal{T}_y^{\mathcal{L}_2,\mathcal{D}_2}$.

b) ν_1 and ν_2 are *similar* if for every atomic p, $\nu_1(p)$ and $\nu_2(p)$ are similar.

Proposition 3.5 Let $(\mathcal{L}_1,\mathcal{D}_1)$ and $(\mathcal{L}_2,\mathcal{D}_2)$ be two logical lattices and suppose that ν_1 and ν_2 are two similar valuations on L_1 and L_2 (respectively). Then for every formula ψ, $\nu_1(\psi)$ and $\nu_2(\psi)$ are similar.

Proof: By an induction on the structure of ψ.[5] □

Proof of Theorem 3.1: We shall denote by m_x some element in $\min_{\leq_c^{\mathcal{L},\mathcal{D}}}\mathcal{T}_x^{\mathcal{L},\mathcal{D}}$ $(x\in\{t,f,\top,\perp\})$, and by $\omega: L\to\{t,f,\top,\perp\}$ the "categorisation" function: $\omega(y)=x$ iff $y\in\mathcal{T}_x$. Also, in the rest of this proof we shall abbreviate $[y]\cap\Omega_{\leq_c^{\mathcal{L},\mathcal{D}}}$ by $[y]$ (thus we shall refer here to subclasses that consist only of elements in $\Omega_{\leq_c^{\mathcal{L},\mathcal{D}}}$).

Claim 3.1-A: If $M\in\,!(\Gamma,\leq_c^{\mathcal{L},\mathcal{D}})$ then for every atom p, $M(p)\in\Omega_{\leq_c^{\mathcal{L},\mathcal{D}}}$.

Proof: Suppose that there is some atom p_0 s.t. $M(p_0)\notin\Omega_{\leq_c^{\mathcal{L},\mathcal{D}}}$. Then, assuming that $M(p_0)\in\mathcal{T}_x$, there is an $m_x\in\min_{\leq_c^{\mathcal{L},\mathcal{D}}}\mathcal{T}_x$ s.t. $m_x<_c^{\mathcal{L},\mathcal{D}}M(p_0)$. Consider the following valuation:

$$N(p)=\begin{cases} m_x & \text{if } p=p_0\\ M(p) & \text{if } p\neq p_0\end{cases}$$

[5] The fact that $\mathcal{D}$ is a *prime* filter is crucial here.

N is similar to M, and so, by Proposition 3.5, N is also a model of Γ. Moreover, $N <_c^{\mathcal{L},\mathcal{D}} M$, thus $M \notin !(\Gamma, \leq_c^{\mathcal{L},\mathcal{D}})$. $\square$

Since $\leq_c^{\mathcal{L},\mathcal{D}}$ is well-founded and since $\mathcal{T}_x$ is nonempty for every $x \in \{t, f, \top, \bot\}$, $\min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_x$ is nonempty as well, and so there is at least one element of the form m_x for every $x \in \{t, f, \top, \bot\}$. Also, it is clear that for every $m_x, m'_x \in \min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_x$, $[m_x] = [m'_x]$ (otherwise either $m_x <_c^{\mathcal{L},\mathcal{D}} m'_x$ or $m_x >_c^{\mathcal{L},\mathcal{D}} m'_x$, and so either $m'_x \notin \min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_x$ or $m_x \notin \min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_x$). It follows, therefore, that there are no more than three equivalence classes in $\Omega_{\leq_c^{\mathcal{L},\mathcal{D}}}$:

$$\min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_t \cup \min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_f \subseteq [t], \quad \min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_\bot \subseteq [m_\bot], \quad \min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_\top \subseteq [m_\top],$$

where $m_\bot$ is some element of $\min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_\bot$, and $m_\top$ is some element of $\min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_\top$. By Definition 2.5, $[t]$ must be a minimal inconsistency class among those in $\Omega_{\leq_c^{\mathcal{L},\mathcal{D}}}$, and $[m_\top]$ must be a maximal one. It follows, then, that the inconsistency classes in $\Omega_{\leq_c^{\mathcal{L},\mathcal{D}}}$ are arranged in one of the following orders:

0. $[t] = [m_\bot] = [m_\top]$,
1. $[t] = [m_\bot] <_c^{\mathcal{L},\mathcal{D}} [m_\top]$
2. $[t] <_c^{\mathcal{L},\mathcal{D}} [m_\bot] = [m_\top]$
3. $[t] <_c^{\mathcal{L},\mathcal{D}} [m_\bot] <_c^{\mathcal{L},\mathcal{D}} [m_\top]$

If the order relation among the inconsistency classes in $\Omega_{\leq_c^{\mathcal{L},\mathcal{D}}}$ corresponds to case i above $(0 \leq i \leq 3)$ we say that the inconsistency order $\leq_c^{\mathcal{L},\mathcal{D}}$ is *of type* i.[6]

Claim 3.1-B: If $\leq_c^{\mathcal{L},\mathcal{D}}$ is an inconsistency order of type i, then for every $m, m' \in \Omega_{\leq_c^{\mathcal{L},\mathcal{D}}}$ we have that $[m] <_c^{\mathcal{L},\mathcal{D}} [m']$ iff $[\omega(m)] <_{c_i}^4 [\omega(m')]$.

Proof: Immediate from the definition of inconsistency order of type i, and the definition of $\leq_{c_i}^4$. $\square$

Claim 3.1-C: If $\leq_c^{\mathcal{L},\mathcal{D}}$ is an inconsistency order of type i in $(\mathcal{L}, \mathcal{D})$, then $\models_c^{\mathcal{L},\mathcal{D}}$ is the same as $\models_{c_i}^4$.

Proof: Suppose that $\Gamma \models_c^{\mathcal{L},\mathcal{D}} \Delta$ but $\Gamma \not\models_{c_i}^4 \Delta$. Then there is a c_i^4-mcm M^4 of Γ s.t. $M^4(\delta) \notin \{t, \top\}$ for every $\delta \in \Delta$. Now, for every atom p let $M^L(p)$ be some element in $\min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_{M^4(p)}$. Thus $\omega \circ M^L = M^4$, and M^L is similar to M^4. By Proposition 3.5, M^L is a model of Γ and it is not a model of any formula in Δ. It remains to show, therefore, that M^L is a c-mcm of Γ in $(\mathcal{L}, \mathcal{D})$ (and so we will have a contradiction to $\Gamma \models_c^{\mathcal{L},\mathcal{D}} \Delta$). Indeed, otherwise by stopperdness there is a c-mcm N^L of Γ s.t. $N^L <_c^{\mathcal{L},\mathcal{D}} M^L$. So for every atom p, $[N^L(p)] \leq_c^{\mathcal{L},\mathcal{D}} [M^L(p)]$, and there is an atom p_0 s.t. $[N^L(p_0)] <_c^{\mathcal{L},\mathcal{D}} [M^L(p_0)]$. Let $N^4 = \omega \circ N^L$. Again, N^4 is similar to N^L, therefore it is a (four-valued) model of Γ. Also, by its definition, for every atom p, $M^L(p) \in \Omega_{\leq_c^{\mathcal{L},\mathcal{D}}}$ and by Claim 3.1-A, $\forall p \; N^L(p) \in \Omega_{\leq_c^{\mathcal{L},\mathcal{D}}}$. Thus, by Claim 3.1-B,

$$[N^4(p)] = [\omega \circ N^L(p)] \leq_{c_i}^4 [\omega \circ M^L(p)] = [M^4(p)].$$

[6] In particular, for every $0 \leq i \leq 3$, the inconsistency order $\leq_{c_i}^4$ in $\mathcal{FOUR}$ is of type i.

Also, by the same claim,

$$[N^4(p_0)] = [\omega \circ N^L(p_0)] <^4_{c_i} [\omega \circ M^L(p_0)] = [M^4(p_0)].$$

It follows that $N^4 <^4_{c_i} M^4$, but this contradicts the assumption that M^4 is a c^4_i-mcm of Γ.

For the converse, suppose that $\Gamma \models^4_{c_i} \Delta$, but $\Gamma \not\models^{\mathcal{L},\mathcal{D}}_c \Delta$. Then there is a c-mcm M^L of Γ in $(\mathcal{L},\mathcal{D})$ s.t. $M^L(\delta) \notin \mathcal{D}$ for every $\delta \in \Delta$. Define, for every atom p, $M^4(p) = \omega \circ M^L(p)$. By the definition of ω, M^4 is similar to M^L and so M^4 is a model of Γ in $\mathcal{FOUR}$, but it is not a model of any formula in Δ. It remains to show, then, that M^4 is a c^4_i-mcm of Γ. Indeed, otherwise there is a model N^4 of Γ s.t. $N^4 <^4_{c_i} M^4$, that is: For every atom p $[N^4(p)] \leq^4_{c_i} [M^4(p)]$, and there is an atom p_0 for which this inequality is strict: $[N^4(p_0)] <^4_{c_i} [M^4(p_0)]$. Now, for every atom p, let $N^L(p)$ be some element in $\min_{\leq^{\mathcal{L},\mathcal{D}}_c} \mathcal{T}_{N^4(p)}$. Thus $\omega \circ N^L = N^4$, and N^L is similar to N^4. By Proposition 3.5 N^L is in particular a model of Γ in $(\mathcal{L},\mathcal{D})$. Moreover, for every atom p,

$$[\omega \circ N^L(p)] = [N^4(p)] \leq^4_{c_i} [M^4(p)] = [\omega \circ M^L(p)].$$

Now, by the definition of N^L we have that for every atom p, $N^L(p) \in \Omega_{\leq^{\mathcal{L},\mathcal{D}}_c}$, and by Claim 3.1-A $M^L(p) \in \Omega_{\leq^{\mathcal{L},\mathcal{D}}_c}$ as well. Hence, by Claim 3.1-B we have that $[N^L(p)] \leq^{\mathcal{L},\mathcal{D}}_c [M^L(p)]$. Similarly,

$$[\omega \circ N^L(p_0)] = [N^4(p_0)] <^4_{c_i} [M^4(p_0)] = [\omega \circ M^L(p_0)]$$

and again this entails that $[N^L(p_0)] <^{\mathcal{L},\mathcal{D}}_c [M^L(p_0)]$. It follows that $N^L <^{\mathcal{L},\mathcal{D}}_c M^L$, but this contradicts the assumption that M^L is a c-mcm of Γ in $(\mathcal{L},\mathcal{D})$.
This concludes the proof of Claim 3.1-C and Theorem 3.1. □

Below are some immediate consequences of Theorem 3.1:

Corollary 3.6 If $(\mathcal{L},\mathcal{D})$ is a finite logical lattice then for every inconsistency order $\leq^{\mathcal{L},\mathcal{D}}_c$ in $(\mathcal{L},\mathcal{D})$ there is an inconsistency order $\leq^4_{c_i}$ in $\mathcal{FOUR}$ s.t. for every finite set Γ of premises and every set Δ of conclusions we have that $\Gamma \models^{\mathcal{L},\mathcal{D}}_c \Delta$ iff $\Gamma \models^4_{c_i} \Delta$.

Proof: The claim immediately follows from Theorem 3.1 once we show stopperdness. In fact, in the proof of Theorem 3.1 we only had to assume that the set of models of the premises is stoppered w.r.t. $\leq^{\mathcal{L},\mathcal{D}}_c$, i.e.: for every $M \in mod(\Gamma)$ either M is a c-mcm of Γ, or there is a c-mcm M' of Γ s.t. $M' <^{\mathcal{L},\mathcal{D}}_c M$. Here, the assumptions on $(\mathcal{L},\mathcal{D})$ and on Γ guarantee the stopperdness of $mod(\Gamma)$. Indeed, let M some a model of Γ. Since L is finite, for every $p \in \mathcal{A}(\Gamma)$[7] there are only finite number of elements that are either $\leq^{\mathcal{L},\mathcal{D}}_c$-smaller than $M(p)$ or $\leq^{\mathcal{L},\mathcal{D}}_c$-incomparable with $M(p)$. Thus, since Γ is finite, the amount of valuations ν s.t. $\forall p \in \mathcal{A}(\Gamma)$ $M(p) \not\leq^{\mathcal{L},\mathcal{D}}_c \nu(p)$ and $\exists p_0 \in \mathcal{A}(\Gamma)$ s.t. $\nu(p_0) <^{\mathcal{L},\mathcal{D}}_c M(p_0)$ is also finite. Hence there is some $\nu_0 \leq^{\mathcal{L},\mathcal{D}}_c M$ s.t. $\nu_0 \in !(\Gamma, \leq^{\mathcal{L},\mathcal{D}}_c)$.
□

Corollary 3.7 Let $\leq^{\mathcal{L},\mathcal{D}}_c$ be an inconsistency order in $(\mathcal{L},\mathcal{D})$ that defines a stoppered relation on $\mathcal{V}^L$. Let also $m_\bot \in \min_{\leq^{\mathcal{L},\mathcal{D}}_c} \mathcal{T}_\bot$ and $m_\top \in \min_{\leq^{\mathcal{L},\mathcal{D}}_c} \mathcal{T}_\top$.

[7]Recall that this means that p is an atomic formulae that appears in some formula of Γ.

a) if $[m_\top] = [t]$, then $\models_c^{\mathcal{L},\mathcal{D}}$ is the same as $\models_{c_0}^4$.

b) if $[m_\top] \neq [t]$ and $[m_\perp] = [t]$, then $\models_c^{\mathcal{L},\mathcal{D}}$ is the same as $\models_{c_1}^4$.

c) if $[m_\top] \neq [t]$ and $[m_\perp] \neq [t]$ and $[m_\top] = [m_\perp]$, then $\models_c^{\mathcal{L},\mathcal{D}}$ is the same as $\models_{c_2}^4$.

d) if $[m_\top] \neq [t]$ and $[m_\perp] \neq [t]$ and $[m_\top] \neq [m_\perp]$, then $\models_c^{\mathcal{L},\mathcal{D}}$ is the same as $\models_{c_3}^4$.

Proof: Follows from the proof of Theorem 3.1. For instance, in terms of that proof, the condition in part (a) assures that $\leq_c^{\mathcal{L},\mathcal{D}}$ is of type 0. Thus $\models_c^{\mathcal{L},\mathcal{D}}$ must be the same as $\models_{c_0}^4$ in this case. Similar considerations hold for the other cases. □

Corollary 3.7 induces a simple algorithm for determining which one of the four-valued consequence relations is the same as a given consequence relation of the form $\models_c^{\mathcal{L},\mathcal{D}}$: Given an inconsistency order $\leq_c^{\mathcal{L},\mathcal{D}}$ in $(\mathcal{L},\mathcal{D})$, choose some $m_\perp \in \min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_\perp$ and $m_\top \in \min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_\top$. If it is true that $[m_\top] = [t]$, then $\models_c^{\mathcal{L},\mathcal{D}}$ is the same as $\models_{c_0}^4$. Otherwise, if $[m_\perp] = [t]$, then $\models_c^{\mathcal{L},\mathcal{D}}$ is the same as $\models_{c_1}^4$. Otherwise, if $[m_\top] = [m_\perp]$, then $\models_c^{\mathcal{L},\mathcal{D}}$ is the same as $\models_{c_2}^4$. Otherwise $\models_c^{\mathcal{L},\mathcal{D}}$ is the same as $\models_{c_3}^4$.

4 Reasoning with $\models_c^{\mathcal{L},\mathcal{D}}$

Our goal in this section is to show that consequence relations of the form $\models_c^{\mathcal{L},\mathcal{D}}$ are particularly suitable for imitating commonsense reasoning. For this we consider some of their useful properties. By Theorem 3.1 and Corollary 3.6, when stopperdness is assumed or the set of premises is finite, it is sufficient to consider only the logical lattices $\mathcal{THREE}$ and $\mathcal{FOUR}$. In what follows we therefore consider the four basic four-valued consequence relations $\models_{c_i}^4$ $(i = 0, \ldots, 3)$. Similar results are easily obtained for the basic three-valued relations $\models_{c_0}^3$ and $\models_{c_1}^3$ (see also the discussion on three-valued consequence relations in Section 5).

Proposition 4.1 Let Γ, Δ be two sets of formulae in Σ.

a) The consequence relations $\models_{c_i}^4$, $0 \leq i \leq 3$, are all different.

b) For every $1 \leq i \leq 3$, if $\Gamma \models_{c_0}^4 \Delta$ then $\Gamma \models_{c_i}^4 \Delta$.

c) No one of $\models_{c_1}^4$, $\models_{c_2}^4$, and $\models_{c_3}^4$, is stronger than the other.

Proof:
a) Consider the set $\Gamma = \{\neg q,\ (p \supset q) \vee (\neg q \supset \neg p),\ (\neg p \supset q) \vee (\neg q \supset p)\}$. Table 3 lists the c_i^4-mcms of Γ. It is easy to verify that for every $0 \leq i \leq 3$ the consequences of Γ w.r.t. $\models_{c_i}^4$ are different. Let $Th_i(\Gamma) = \{\psi \mid \Gamma \models_{c_i}^4 \psi\}$. Then from Table 3 it follows that $Th_0(\Gamma) \subseteq Th_2(\Gamma) \subseteq Th_3(\Gamma) \subseteq Th_1(\Gamma)$. Moreover, $q \supset p \in Th_1(\Gamma) \setminus Th_3(\Gamma)$, $p \supset q \in Th_3(\Gamma) \setminus Th_2(\Gamma)$, and $q \supset (p \vee \neg p) \in Th_2(\Gamma) \setminus Th_0(\Gamma)$, so the inclusions are proper.

b) Obvious.

Table 3: The c_i^4-mcms of Γ

	p	q	c_0^4-mcms	c_1^4-mcms	c_2^4-mcms	c_3^4-mcms
M_1	$\bot$	f	+	+	+	+
M_2	$\top$	f	+	–	+	–
M_3	t	$\top$	+	–	+	+
M_4	f	$\top$	+	–	+	+
M_5	$\bot$	$\top$	+	–	–	–
M_6	$\top$	$\top$	+	–	–	–

c) In part (a) we have considered an example in which $Th_2(\Gamma) \subset Th_3(\Gamma) \subset Th_1(\Gamma)$. On the other hand, $p \vee \neg p \in Th_2(\emptyset)$ and $p \vee \neg p \in Th_3(\emptyset)$, while $p \vee \neg p \notin Th_1(\emptyset)$. It remains to show, then, that $\models^4_{c_3}$ is not stronger than $\models^4_{c_2}$. For that consider the following set: $\Gamma' = \{p,\ (\neg p \supset q) \supset q,\ q \supset \neg q,\ \neg q \supset q\}$. The only c_2^4-mcm of Γ' is $M_1(p) = t$, $M_1(q) = \top$, while the c_3^4-mcms of Γ' are M_1 and $M_2(p) = \top$, $M_2(q) = \bot$. Thus, e.g., $\Gamma' \models^4_{c_2} q$ while $\Gamma' \not\models^4_{c_3} q$. In this case, therefore, $Th_3(\Gamma') \subset Th_2(\Gamma')$. $\Box$

Next we consider some more general properties of $\models^4_{c_i}$ $(0 \leq i \leq 3)$. In what follows we shall write $\models^4_c$ for referring to any of $\models^4_{c_i}$, $0 \leq i \leq 3$.

4.1 Oscillation between classical and paraconsistent inferences

A desirable property of formalisms for managing inconsistent information is that they will to be able to draw classical conclusions from (classically) consistent theories, and will not "explode" the set of conclusions when the theory becomes inconsistent. Batens [7] describes this property as an "oscillation" between some lower limit (paraconsistent) logic and an upper limit (classical) logic: The rules of the lower limit logic are unconditionally correct, while supplementary rules of the upper limit logic are correct under certain conditions that depend on the premises. In this section we show that this is the case in our framework.

Proposition 4.2 $\models^4_c$ is paraconsistent.

Proof: Indeed, $p, \neg p \not\models^4_c q$. To see that consider a valuation ν, for which $\nu(p) = \top$ and $\nu(q) = f$. $\Box$

In what follows we denote by $\models^2$ the classical consequence relation.

Proposition 4.3 If $\Gamma \models^4_c \Delta$ then $\Gamma \models^2 \Delta$.

Proof: Let M be a classical model of Γ. Since the set $\{t, f\}$ is closed under the corresponding operations, there is no difference between viewing M as a valuation in $\mathcal{FOUR}$ and viewing it as a valuation in $\{t, f\}$. Hence M is also a model of Γ in $\mathcal{FOUR}$. Now, since M assigns only classical truth values to the atomic formulae, M must be a c-mcm of Γ in $\mathcal{FOUR}$. Since

$\Gamma \models^4_c \Delta$, there is some $\delta \in \Delta$ s.t. $M(\delta) \in \{t, \top\}$. But we also know that $M(\delta) \in \{t, f\}$, and so necessarily $M(\delta) = t$. It follows that M is a classical model of some formula in Δ, and so $\Gamma \models^2 \Delta$. $\square$

The converse of Proposition 4.3 is not true in general. For instance, excluded middle is not valid w.r.t. $\models^4_{c0}$ and $\models^4_{c1}$. However, with respect to the other basic four-valued consequence relations the converse of Proposition 4.3 does hold.

Proposition 4.4 Let Γ be a classically consistent theory. Then for every formula ψ we have that $\Gamma \models^2 \Delta$ iff $\Gamma \models^4_{c_2} \Delta$ iff $\Gamma \models^4_{c_3} \Delta$.

Proof: Immediately follows from the fact that the set of the c^4_2-mcms and the set of the c^4_3-mcms of a classically consistent theory Γ are the same as the set of the classical models of Γ. $\square$

By Propositions 4.2 and 4.4 we obtain the following important property of (any consequence relation of the form $\models^{\mathcal{L},\mathcal{D}}_c$ that is equivalent to) $\models^4_{c_2}$ and $\models^4_{c_3}$:

Corollary 4.5 $\models^4_{c_2}$ and $\models^4_{c_3}$ are the same as classical logic w.r.t. consistent theories, and are not trivial w.r.t. inconsistent theories.

4.2 Nonmonotonicity and plausibility

Proposition 4.6 $\models^4_{c0}$ is a monotonic consequence relation, while $\models^4_{c_i}$, $i = 1, 2, 3$, are nonmonotonic.

Proof: For the first part, note that $\models^4_{c0}$ is in fact the same as $\models^4$, which is clearly monotonic. For the other part, consider $\Gamma = \{p, \neg p \vee q\}$. Since $M(p) = t$, $M(q) = t$ is the only c^4_i-mcm of Γ for $i = 1, 2, 3$, it follows that $\Gamma \models^4_{c_i} q$ $(i = 1, 2, 3)$. However, as in the proof of Proposition 4.2, it is easy to see that $\Gamma, \neg p \not\models^4_{c_i} q$ for $i = 1, 2, 3$. $\square$

Note that by Corollary 3.7 and Proposition 4.6 it follows that $\models^{\mathcal{L},\mathcal{D}}_c$ is nonmonotonic iff $[t] \cap \min_{\leq^{\mathcal{L},\mathcal{D}}_c} \mathcal{T}_\top = \emptyset$. It follows that unless the inconsistency order under consideration is degenerated, the consequence relation that is based on it is nonmonotonic. Thus, most of the relations of the form $\models^{\mathcal{L},\mathcal{D}}_c$ are not consequence relations in the standard sense of Tarski and Scott. In such cases it is usual to require weaker conditions:

Proposition 4.7 $\models^4_c$ is a plausibility logic in the sense of Lehmann [16]. I.e., the following properties are satisfied:

Inclusion: $\Gamma, \psi \models^4_c \psi$.

Right Monotonicity: If $\Gamma \models^4_c \Delta$, then $\Gamma \models^4_c \psi, \Delta$.

Cautious Left Monotonicity: If $\Gamma \models^4_c \psi$ and $\Gamma \models^4_c \Delta$, then $\Gamma, \psi \models^4_c \Delta$. [8]

[8] This rule was first proposed in [13].

Cautious Cut: If $\Gamma, \psi \models^4_c \Delta$ and $\Gamma \models^4_c \psi, \Delta$, then $\Gamma \models^4_c \Delta$.

Proof: Inclusion and Right Monotonicity immediately follow from the definition of $\models^4_c$. For Cautious Left Monotonicity, assume that $\Gamma \models^4_c \psi$, and $\Gamma \models^4_c \Delta$. Let M be some c-mcm of $\Gamma \cup \{\psi\}$. In particular, M is a model of Γ. Moreover, it must be a c-mcm of Γ as well, since otherwise there would have been some $N \in mod(\Gamma)$ s.t. $N <^4_c M$. Since $\Gamma \models^4_c \psi$, this N would have been a model of $\Gamma \cup \{\psi\}$ which is strictly less inconsistent than M. Hence M cannot be a c-mcm of $\Gamma \cup \{\psi\}$, with a contradiction to the choice of M. Therefore, M is a c-mcm of Γ. Now, since $\Gamma \models^4_c \Delta$, M is a model of some $\delta \in \Delta$. Hence $\Gamma, \psi \models^4_c \Delta$.

It remains to show Cautious Cut: Let M be a c-mcm of Γ. Suppose, for a contradiction, that $\Gamma, \psi \models^4_c \Delta$ and $\Gamma \models^4_c \psi, \Delta$, but $M(\delta) \notin \mathcal{D}$ for every $\delta \in \{t, \top\}$. Since $\Gamma \models^4_c \psi, \Delta$, necessarily $M(\psi) \in \{t, \top\}$, and so M is a model of $\Gamma \cup \{\psi\}$. Moreover, M must be a c-mcm model of $\Gamma \cup \{\psi\}$, since any other model of this set that is strictly smaller than M w.r.t. $\leq^4_c$ must be in particular a model of Γ, which is $\leq^4_c$-smaller than M. Now, $\Gamma, \psi \models^4_c \Delta$, therefore $M(\delta) \in \{t, \top\}$ for some $\delta \in \Delta$ – a contradiction. $\Box$

4.3 Rationality

In [18] Lehmann and Magidor consider some properties of nonmonotonic reasoning that a "rational" nonmonotonic consequence relation should satisfy. One property that is considered as particularly important is motivated by the following example: Suppose that all we know about normal birds is that they can fly. Then, unless anything else is known about red birds, it seems reasonable to assume that (normal) red birds can fly as well. This property is sometimes called the "irrelevance problem": A reasoner does not have to retract any previous conclusion when learning about a new fact that has no influence on the set of premises. Consequence relations that satisfy this property are called *rational*. As the next proposition shows, the consequence relations $\models^4_{c_i}$ are rational:

Proposition 4.8 If $\Gamma \models^4_c \Delta$ and $\mathcal{A}(\Gamma \cup \Delta) \cap \mathcal{A}(\Lambda) = \emptyset$, then $\Gamma, \Lambda \models^4_c \Delta$.

Proof: If $\Gamma, \Lambda \not\models^4_c \Delta$, then there is an $M \in\, !(\Gamma \cup \Lambda, \leq^4_c)$ s.t. for every $\delta \in \Delta$, $M(\delta) \notin \{t, \top\}$. Let m be some $\leq^4_c$-minimal element. Consider the following valuation:

$$N(p) = \begin{cases} M(p) & \text{if } p \in \mathcal{A}(\Gamma \cup \Delta) \\ m & \text{otherwise} \end{cases}$$

Clearly, N is a model of Γ and for every $\delta \in \Delta$, $N(\delta) \notin \{t, \top\}$. Since $\Gamma \models^4_c \Delta$, N cannot be a c-mcm of Γ, and so there is a model N' of Γ s.t. $N' <^4_c N$. By the definition of N, there is some $p_0 \in \mathcal{A}(\Gamma \cup \Delta)$ s.t. $N'(p_0) <^4_c N(p_0)$. Now, consider the following valuation:

$$M'(p) = \begin{cases} N'(p) & \text{if } p \in \mathcal{A}(\Gamma \cup \Delta) \\ M(p) & \text{otherwise} \end{cases}$$

Clearly, $M' <^4_c M$, and since M' is the same as N' on $\mathcal{A}(\Gamma)$, M' is also a model of Γ. Moreover, using the facts that $\mathcal{A}(\Gamma \cup \Delta) \cap \mathcal{A}(\Lambda) = \emptyset$ and that M is a model of Λ, it follows that M' is also a model of Λ. Hence M' is a model of $\Gamma \cup \Lambda$, which is strictly $\leq_c^4$-smaller than M, but this is a contradiction to the choice of M. $\Box$

Note: In order to assure rationality, Lehmann and Magidor introduced the following inference rule (see [18]):

$$\textit{rational monotonicity:} \qquad \text{if } \Gamma \mid\!\sim \psi \text{ then } \Gamma, \phi \mid\!\sim \psi, \text{ unless } \Gamma \mid\!\sim \neg\phi.$$

Rational monotonicity might be considered as a requirement that is too strong for assuring rationality, and there are many general patterns of nonmonotonic reasoning that do not satisfy rational monotonicity. For instance, the following example shows that rational monotonicity is not sound w.r.t. $\models_{c_1}^4$: $p,\ q \supset \neg p \models_{c_1}^4 \neg p \supset \neg q$ and $p,\ q \supset \neg p \not\models_{c_1}^4 \neg q$, but $p,\ q,\ q \supset \neg p \not\models_{c_1}^4 \neg p \supset \neg q$.

4.4 Adaptivity

Consider the following set of formulae: $\Gamma_1 = \{p, \neg p, \neg p \vee q\}$. Since $\neg p$ is true in Γ_1, so is $\neg p \vee q$ (even if q is false), and so a plausible inference mechanism should not apply here the Disjunctive Syllogism to p and $\neg p \vee q$. Thus, plausible paraconsistent systems should not validate the Disjunctive Syllogism in any case. On the other hand, in the case of $\Gamma_2 = \{p, \neg p, r, \neg r \vee q\}$, applying the Disjunctive Syllogism to r and $\neg r \vee q$ may be justified by the fact that the subset of formulae to which the Disjunctive Syllogism is applied should not be affected by the inconsistency, therefore inference rules that are classically valid can be applied to it.

This is the basic idea behind Baten's *inconsistency adaptive logics* [6, 7, 8]. Such logics are capable of handling theories with contradictions in a nontrivial way, but presuppose a consistency of all sentences 'unless and until proven otherwise'. By interpreting a theory 'as consistently as possible', they *adapt* to the *specific* inconsistencies that occur in it.

As Proposition 4.9 below shows, the consequence relations $\models_{c_2}^4$ and $\models_{c_3}^4$ are adaptive; If one can distinguish between a consistent part and an inconsistent part of a given theory, then every assertion that is not related to the inconsistent part, and which classically follows from the consistent part, is also a $\models_{c_i}^4$-consequence ($i = 2, 3$) of the whole theory.

Proposition 4.9 Let $\Gamma = \Gamma' \cup \Gamma''$ be a set of formulae in Σ s.t. $\mathcal{A}(\Gamma') \cap \mathcal{A}(\Gamma'') = \emptyset$. If Γ' is classically consistent, then for every set Δ s.t. $\mathcal{A}(\Delta) \cap \mathcal{A}(\Gamma'') = \emptyset$, the fact that $\Gamma' \models^2 \Delta$ entails that $\Gamma \models_{c_2}^4 \Delta$ and $\Gamma \models_{c_3}^4 \Delta$.

Proof: We show here the case of $\models_{c_2}^4$; The argument for $\models_{c_3}^4$ is the same. Suppose that $\Gamma' \models^2 \Delta$. By Proposition 4.4 $\Gamma' \models_{c_2}^4 \Delta$, and by Proposition 4.8, since $\mathcal{A}(\Gamma' \cup \Delta) \cap \mathcal{A}(\Gamma'') = \emptyset$, we have that $\Gamma \models_{c_2}^4 \Delta$. $\Box$

5 Comparative study

Many systems for preferential reasoning have been considered in the literature. Here we briefly mention some closely related ones. First, as we have already noted, $\models_{c_0}^3$ corresponds to one of the basic three-valued paraconsistent logics LP [14], and $\models^4_{c_0}$ corresponds to the four-valued logic of Belnap [9, 10]. Also, as it is shown in Proposition 4.4, $\models^4_{c_2}$ and $\models^4_{c_3}$ are the same as the classical consequence relation w.r.t. theories that are classically consistent.

Another closely related formalism is Priest's three-valued logic of minimally inconsistent models [23, 24]. The consequence relation $\models^3_{\mathrm{LPm}}$ of the resulting logic, LPm, is defined in $\mathcal{THREE}$ as follows: $\Gamma \models^3_{\mathrm{LPm}} \Delta$ iff every model of Γ that assign $\top$ only to some minimal set of atomic formulae is a model of a formulae in Δ. It follows that this consequence relation is actually the same as the one denoted here by $\models^3_{c_1}$.[9] The following proposition indicates a possible reduction of our four-valued consequence relations to Priest's LPm:

Proposition 5.1 Suppose that $\mathcal{A}(\Gamma \cup \Delta) = \{p_1, p_2, \ldots\}$. Then, for any $1 \leq i \leq 3$, $\Gamma \models^3_{\mathrm{LPm}} \Delta$ iff $\Gamma, p_1 \vee \neg p_1, p_2 \vee \neg p_2, \ldots \models^4_{c_i} \Delta$.

Proof: The three-valued models of Γ are the same as the four-valued models of $\Gamma \cup \{p_1 \vee \neg p_1, p_2 \vee \neg p_2, \ldots\}$. Since each one of these models assigns to the atomic formulae in $\mathcal{A}(\Gamma \cup \{\psi\})$ values from $\{t, f, \top\}$, the LPm-models of Γ are the same as the c_i-mcms of $\Gamma \cup \{p_1 \vee \neg p_1, p_2 \vee \neg p_2, \ldots\}$ for $i = 1, 2, 3$. $\Box$

In the language without $\supset$ (which is the language that is considered in [23, 24]), Priest's logic is the same as $\models^4_{c_2}$ and $\models^4_{c_3}$:

Proposition 5.2 Let Γ, Δ be two sets of formulae in the language of $\{\neg, \vee, \wedge, t, f\}$. Then $\Gamma \models^3_{\mathrm{LPm}} \Delta$ iff $\Gamma \models^4_{c_2} \Delta$ iff $\Gamma \models^4_{c_3} \Delta$.

Proposition 5.2 follows from the next proposition, which implies that in the language of $\{\neg, \vee, \wedge, t, f\}$, the consequence relations $\models^4_{c_2}$ and $\models^4_{c_3}$ are in fact three-valued:

Proposition 5.3 Let Γ be a set of formulae in the language of $\{\neg, \vee, \wedge, t, f\}$. Then:

a) If M is a c_2-mcm of Γ then there is no formula ψ s.t. $M(\psi) = \bot$.

b) If M is a c_3-mcm of Γ then there is no formula ψ s.t. $M(\psi) = \bot$.

Proof: We only show part (a); The proof of part (b) is similar. First, note that $\{t, f, \top\}$ is closed under $\neg, \vee$ and $\wedge$. Now, let M be some c_2-mcm of Γ. Define a transformation $g : \{t, f, \top, \bot\} \to \{t, f, \top\}$ as follows: $g(\bot) = t$, and $g(x) = x$ otherwise. As it is easily verified, for every formulae ψ in the language of $\{\neg, \vee, \wedge, t, f\}$ s.t. $M(\psi)$ is designated, $g \circ M(\psi)$ is designated as well. It follows that $g \circ M$ is also a model of Γ. Since $g \circ M \leq^4_{c_2} M$, necessarily $g \circ M = M$. $\Box$

[9] In fact, in [23, 24] the consequence relation under consideration is single-conclusioned. We use here the obvious extension to the multiple-conclusion case.

Another formalism, which is a particular case of the one considered here, was introduced in [1, 2]. This formalism is based on the notion of *inconsistency sets* in logical lattices.[10] Intuitively, an inconsistency set $\mathcal{I}$ in a logical lattice $(\mathcal{L},\mathcal{D})$ contains the elements of L that represent inconsistent knowledge or belief. Formally, a subset $\mathcal{I}$ of a lattice L is called an inconsistency set if for every $x \in L$ the following two conditions hold: (a) $x \in \mathcal{I}$ iff $\neg x \in \mathcal{I}$, and (b) $x \in \mathcal{D} \cap \mathcal{I}$ iff $x \in \mathcal{T}_{\top}^{\mathcal{L},\mathcal{D}}$.

Let $I(\nu,\mathcal{I})$ denote the set of atomic formulae that are assigned an inconsistent value by a valuation ν. I.e.,

$$I(\nu,\mathcal{I}) = \{p \mid p \text{ is atomic and } \nu(p) \in \mathcal{I}\}.$$

Given an inconsistency set $\mathcal{I}$ in a logical lattice $(\mathcal{L},\mathcal{D})$, a valuation ν_1 is *more consistent* (w.r.t. $\mathcal{I}$) than a valuation ν_2 (notation $\nu_1 <_{\mathcal{I}}^{\mathcal{L},\mathcal{D}} \nu_2$) if $I(\nu_1,\mathcal{I}) \subset I(\nu_2,\mathcal{I})$. A valuation ν is an *$\mathcal{I}$-most consistent model* of Γ ($\mathcal{I}$-mcm of Γ, for short), if $\nu \in \mathit{mod}(\Gamma)$ and there is no model of Γ which is more consistent than ν. Finally, $\Gamma \models_{\mathcal{I}}^{\mathcal{L},\mathcal{D}} \Delta$ if every $\mathcal{I}$-mcm of Γ is a model of some formulae in Δ.

As the following propositions show, the notion of inconsistency orders is a refinement of the notion of inconsistency sets:

Proposition 5.4 Let $\mathcal{I}$ be an inconsistency set in $(\mathcal{L},\mathcal{D})$. The order relation $\leq_{\mathcal{I}}^{\mathcal{L},\mathcal{D}}$, defined on L by $x \leq_{\mathcal{I}}^{\mathcal{L},\mathcal{D}} y$ if $x \notin \mathcal{I}$ and $y \in \mathcal{I}$, is an inconsistency order.

Proof: It is easy to verify that $\leq_{\mathcal{I}}^{\mathcal{L},\mathcal{D}}$ satisfies all the conditions in Definition 2.5. $\square$

Proposition 5.5 Let $(\mathcal{L},\mathcal{D})$ be a logical lattice, and let $\mathcal{I} = \mathcal{T}_{\top}$. Then for every inconsistency order $\leq_c^{\mathcal{L},\mathcal{D}}$ in $(\mathcal{L},\mathcal{D})$ there is an "intermediate" inconsistency level $[i]$ s.t. for every $x_c \in [i]$ and every $y \in L$, if $y >_c^{\mathcal{L},\mathcal{D}} x_c$ then $y \in \mathcal{I}$, and if $y <_c^{\mathcal{L},\mathcal{D}} x_c$ then $y \notin \mathcal{I}$.

Proof: Let $[i] = \min_{\leq_c^{\mathcal{L},\mathcal{D}}}\{[x] \mid \exists y \in [x] \text{ s.t. } y \in \mathcal{T}_{\top}\}$.[11] This definition entails the second part of the claim, since if $y <_c^{\mathcal{L},\mathcal{D}} x_c$ for some $y \in L$ and $x_c \in [i]$, then $[y] < [i]$, and so there is no element in $[y]$ (especially y itself) that belongs to $\mathcal{I}$. For the other part, let $x_c \in [i]$ and let $y \in L$ s.t. $y >_c^{\mathcal{L},\mathcal{D}} x_c$. By the definition of $[i]$, there is some $x' \in [x_c]$ s.t. $x' \in \mathcal{T}_{\top}$. In particular, x_c and x' are either equal or $\leq_c^{\mathcal{L},\mathcal{D}}$-incomparable, and since $\leq_c^{\mathcal{L},\mathcal{D}}$ is modular, necessarily $y >_c^{\mathcal{L},\mathcal{D}} x'$. By Definition 2.5(b), $y \in \mathcal{T}_{\top}$ as well. $\square$

Corollary 5.6 The family of consequence relations of the form $\models_c^{\mathcal{L},\mathcal{D}}$ strictly contains the family of the consequence relations of the form $\models_{\mathcal{I}}^{\mathcal{L},\mathcal{D}}$.

Proof: Follows from the fact that in terms of Proposition 5.4, $\models_{\mathcal{I}}^{\mathcal{L},\mathcal{D}}$ is the same as $\models_c^{\mathcal{L},\mathcal{D}}$ where $\leq_c^{\mathcal{L},\mathcal{D}}$ is $\leq_{\mathcal{I}}^{\mathcal{L},\mathcal{D}}$. $\square$

[10] The algebraic structures considered in [1, 2] are logical bilattices. This is a particular family of logical lattices.

[11] Note that since $\leq_c^{\mathcal{L},\mathcal{D}}$ is well-founded, $[i]$ cannot be empty.

6 Conclusion

We have introduced a general and simple formalism for reasoning with uncertainty. The underlying concept behind it was the *rationality* of the consequences that it allows. Although many real-life inferences turn out to be wrong, especially in the presence of incomplete or inconsistent information, we always expect them to be "rational". A variety of properties were considered for supporting the plausibility of the reasoning process introduced here, and for showing that it indeed provides "rational" consequences from data that might be incomplete or inconsistent. In addition, we have provided a characterization result that shows that the consequence relations considered here can be represented in Belnap's four-valued algebraic structure, $\mathcal{FOUR}$. This is another evidence for the central role $\mathcal{FOUR}$ has among (logical) lattices; In many cases only a four-valued lattice is sufficient for constructing a "robust" framework for reasoning with uncertainty (see also [3]).

Acknowledgement

I would like to thank Arnon Avron for the helpful discussions on the topics of this paper. This work is supported by the Visiting Postdoctoral Fellowship FWO-Vlaanderen.

References

[1] O.Arieli, A.Avron. *Logical bilattices and inconsistent data.* Proc. LICS'94, pages 468–476, IEEE Press, 1994.

[2] O.Arieli, A.Avron. *Reasoning with logical bilattices.* Journal of Logic, Language, and Information, Vol.5, No.1, pages 25–63, 1996.

[3] O.Arieli, A.Avron. *The value of the four values.* Artificial Intelligence, Vol.102, No.1, pages 97–141, 1998.

[4] O.Arieli, A.Avron. *Nonmonotonic and paraconsistent reasoning: From basic entailments to plausible relations.* Proc. Ecsqaru'99, LNAI No.1638 (A.Hunter, S.Parsons - Eds.), Springer Verlag, pages 11–22, 1999.

[5] A.Avron. *Simple consequence relations.* Journal of Information and Computation, Vol.92, pages 105–139, 1991.

[6] D.Batens. *Dynamic dialectical logics.* Paraconsistent Logic. Essay on the Inconsistent (G.Priest, R.Routely, J.Norman, editors), pages 187-217, Philosophia Verlag, 1989.

[7] D.Batens. *Inconsistency-adaptive logics.* Logic at Work (E.Orlowska, editor), Physica Verlag, pages 445-472, 1998.

[8] D.Batens. *A survey on inconsistency-adaptive logics.* In print.

[9] N.D.Belnap. *A useful four-valued logic.* Modern Uses of Multiple-Valued Logic (G.Epstein, J.M.Dunn, editors), Reidel Publishing Company, Boston, pages 7–37, 1977.

[10] N.D.Belnap. *How computer should think.* Contemporary Aspects of Philosophy (G.Ryle, editor), Oriel Press, pages 30–56, 1977.

[11] N.C.A.da-Costa. *On the theory of inconsistent formal systems.* Notre Damm Journal of Formal Logic, Vol.15, pages 497–510, 1974.

[12] D.Dubios, J.Lang, H.Prade. *Possibilistic logic.* Handbook of Logic in Artificial Intelligence and Logic Programming (D.Gabbay, C.Hogger, J.Robinson, editors), Oxford Science Publications, pages 439–513, 1994.

[13] D.M.Gabbay. *Theoretical foundation for non-monotonic reasoning in expert systems.* Proc. of the NATO Advanced Study Inst. on Logic and Models of Concurrent Systems (K.P.Apt, editor), Springer-Verlag, pages 439–457, 1985.

[14] S.C.Kleene. *Introduction to metamathematics.* Van Nostrand, 1950.

[15] S.Kraus, D.Lehmann, M.Magidor. *Nonmonotonic reasoning, preferential models and cumulative logics.* Artificial Intelligence, Vol.44, No.1–2, pages 167–207, 1990.

[16] D.Lehmann. *Plausibility logic.* Proc. CSL'91, Springer-Verlag, pages 227–241, 1992.

[17] D.Lehmann. *Nonmonotonic logics and semantics.* Technical Report TR-98-6, Institute of Computer Science, Hebrew University, 1998.

[18] D.Lehmann, M.Magidor. *What does a conditional knowledge base entail?*. Artificial Intelligence, Vol.55, pages 1–60, 1992.

[19] D.Makinson. *General theory of cumulative inference*. Non-Monotonic Reasoning (M.Reinfrank, editor), Springer-Verlag, LNAI No.346, pages 1–18, 1989.

[20] D.Makinson. *General patterns in nonmonotonic reasoning*. Handbook of Logic in Artificial Intelligence and Logic Programming, Vol.3 (D.Gabbay, C.Hogger, J.Robinson, editors), Oxford Science Pub., pages 35–110, 1994.

[21] J.McCarthy. *Circumscription – A form of non monotonic reasoning.* Artificial Intelligence, Vol.13, No.1–2, pages 27–39, 1980.

[22] J.Pearl. *Reasoning under uncertainty.* Annual Review of Computer Science, Vol.4, pages 37–72, 1989.

[23] G.Priest. *Reasoning about truth.* Artificial Intelligence, Vol.39, pages 231–244, 1989.

[24] G.Priest. *Minimally Inconsistent LP.* Studia Logica, Vol.50, pages 321–331, 1991.

[25] Y.Shoham. *Reasoning about change: Time and causation from the standpoint of artificial intelligence.* MIT Press, 1988.

[26] V.S.Subrahmanian. *Mechanical proof procedures for many valued lattice-based logic programming.* Journal of Non-Classical Logic, Vol.7, pages 7–41, 1990.

[27] V.S.Subrahmanian. *Paraconsistent disjunctive deductive databases.* Proc. 20th Int. Symp. on Multiple-Valued Logic, IEEE Press, pages 339–345, 1990.